Steady state free thermal convection of liquid in a saturated permeable medium

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SUMMARY

The partial differential equations which describe steady flow of fluid in saturated homogeneous permeable solid material under non-isothermal conditions are stated. From these are derived the equations for flow of liquid (in particular, water) using suitable approximations and making use of empirical laws when necessary.

It is then postulated that the only 'ponderomotive' (i.e. mass-moving) forces present are those due to thermal expansion effects. Free convection results. An approximate solution of the equations is attempted for plane flow by means of classical perturbation methods, the temperature and stream-function variables being represented by power series in a convection parameter proportional to the Rayleigh number.

A numerical example of the method, with boundary conditions based on a geothermal area at Wairakei, New Zealand, is given. The results show features which are in fair agreement with temperature measurements made in the area, and it appears that the convection parameter η is of the order of 10.

1. INTRODUCTION

In some situations associated with geothermal activity, it is possible that the flow of ground water of meteoric origin is influenced by convection currents due to differential heating. Heat is transported both by the convecting liquid and by thermal conduction through the saturated permeable earth. When an area of thermal activity is under investigation for its potentialities as a power source or for other reasons, the relation between ground-water movement and temperature distribution is of practical value, since the latter can be measured relatively easily by means of exploratory boreholes.

The aim of the present study is to deduce approximate solutions for the flow field and temperature distribution when heat conduction and free convection of water in the permeable material are important. As would be expected, the problem is closely related to the problem of free convection in a viscous fluid (Goldstein 1938; Batchelor 1954), in which the appropriate convection parameter is the Rayleigh number, which, in the liquid case, is

$$R = \frac{\alpha(T_1 - T_0)gd^3}{\kappa\nu_0}. \quad (1)$$

Here T_0, T_1 are two reference temperatures on the absolute scale, α is the coefficient of linear thermal volume expansion of the liquid, g is the acceleration of gravity, d is a representative linear dimension, κ is the thermal diffusivity of the liquid at temperature T_0, and ν_0 is its kinematic viscosity at T_0.

When the liquid flows slowly through permeable material, motion is resisted according to the law of Darcy (1856), which states that the hydraulic gradient is proportional to the fluid velocity and to its viscosity, and is inversely proportional to the permeability. In the equation of motion, the Darcy resistance term replaces the Navier–Stokes viscosity term. The corresponding convection parameter η can be defined by

$$\eta = \frac{k\alpha(T_1 - T_0)gd}{(K_m/K_w)\kappa\nu_0} = \frac{N(\Delta/d)^2}{K_m/K_w} R, \tag{2}$$

where K_m/K_w is the ratio of the thermal conductivity of the solid–liquid mixture to the conductivity of the liquid, and κ is, as before, the thermal diffusivity of the liquid. The permeability of the solid medium is $k = N\Delta^2$, Δ being a linear dimension proportional to the particle size, and N a dimensionless constant dependent upon the geometrical shape of the particles.

Usually the ratio η/R is very small, as the permeable solid offers considerable resistance to the flow; and large temperature gradients are necessary before appreciable heat transport by convection is observed. This suggests the use of series expansions of the dependent variables (temperature and stream function) in powers of the parameter η, analogous to the known technique of expansion in powers of the Rayleigh number as used in certain problems of free convection in fluids. The method has limited usefulness in the latter case, since the series are convergent only for small values of R (Batchelor 1954). However, when the heat-transfer process is dominated by conduction, as in the case considered here, the expansion method should prove valid for a moderately large range of temperature differences.

It will be assumed that the permeable solid medium is homogeneous and isotropic in its physical properties, including fluid permeability and thermal conductivity, both of which will be assumed to stay approximately constant with changes in temperature.

Because of the wide temperature range employed, it is necessary to adopt certain empirical results in order to describe the density and viscosity of the liquid (water). As usual, the effect of pressure changes upon the density will be neglected. It remains to describe the effect of thermal volume expansion, which follows a non-linear law in the case of water. For convenience, the approximate empirical expression will be taken in the form of a polynomial relating density ρ and temperature τ (°C), viz.

$$\rho = 0{\cdot}9969\{1 - 3{\cdot}17 \times 10^{-4}(\tau - 25) - 2{\cdot}56 \times 10^{-6}(\tau - 25)^2\}\,\mathrm{gm\,cm^{-3}}. \tag{3}$$

Comparison with values of the density ρ_{sat} for saturated water tabulated by Dorsey (1940) shows that the density difference $(\rho - 0{\cdot}9969)$ is given to an accuracy of 2% in the range 25° C to 250° C. For a lesser temperature range, a linear expansion law might prove to be adequate.

Changes of liquid viscosity with pressure changes will be neglected (Dorsey 1940). However, the liquid viscosity is a rapidly-varying function of temperature which requires to be approximated analytically. The exponential expression due to Andrade (1934) suffices for most liquids, but it is not very tractable and water does not conform to that law. Consequently, for water the approximate empirical expression

$$\nu = 0{\cdot}332(\tau + 13)^{-1}\mathrm{cm}^2\,\mathrm{sec}^{-1} \tag{4}$$

for the kinematic viscosity ν, based on tables (Dorsey 1940), will be adopted. In the range 15° C to 225° C, the expression (4) is accurate to within 5%.

2. Steady-state differential equations

Let the temperature field be continuous throughout a given region of the fluid-saturated homogeneous permeable medium, and let steady-state conditions exist, so that all partial time derivatives are zero.

As usual, the steady-state equation of continuity (Hubbert 1940) is

$$\nabla\,.\,(\rho\mathbf{q}) = 0, \tag{5}$$

where ∇ is the Laplacian operator, ρ is the fluid density, and $\mathbf{q}$ is the flow vector expressed in units of fluid volume crossing unit area in unit time. (It is to be noted that $\mathbf{q}$ is related to the fluid particle velocity vector $\mathbf{v}$ by $\mathbf{q} = \epsilon\mathbf{v}$, where ϵ is the porosity of the medium.)

The linear law of motion due to Darcy (1856) can be written as

$$\frac{1}{\rho}\nabla p - \mathbf{g} + \frac{1}{k}\nu\mathbf{q} = -\epsilon^{-2}\mathbf{q}\,.\,\nabla\mathbf{q}, \tag{6}$$

if p, ν are the pressure and kinematic viscosity of the fluid, $\mathbf{g}$ is the vector of gravitational acceleration, and k is the permeability of the solid medium. The right-hand side of (6) consists of an inertial term which is negligible for the very low Reynolds numbers considered here. Although this term is retained in (6) in order that the conventional derivation of the energy equation (7) should be applicable, it will be neglected in subsequent sections of the paper.

When non-isothermal conditions apply, both ρ and ν will be one-valued continuous functions of the absolute temperature T. The assumption will be made here that equation (6) still holds under these conditions.

An equation for the flow of energy can be derived also by following closely the method for viscous fluids (Goldstein 1938, ch. 14), with the equation (6) replacing the usual Navier–Stokes equation. It is readily shown that the steady-state differential equation of energy transport is

$$Jc\rho\mathbf{q}.\nabla T + p\nabla.\mathbf{q} - \frac{1}{k}\rho\nu\mathbf{q}^2 = J\nabla.(K_m\nabla T) \tag{7}$$

in which K_m is the thermal conductivity of the fluid-saturated permeable medium (assumed constant hereafter), J is the mechanical equivalent of heat, and c is the specific heat per unit mass of the fluid at constant volume. Equation (7) differs from the energy equation of a viscous fluid only in the form of the viscous dissipation term and in the substitution of the flow vector $\mathbf{q}$ for the fluid particle velocity vector.

3. Approximate equations for liquid flow

Now let the fluid be a liquid, assumed incompressible, whose density depends on temperature according to the law

$$\begin{aligned}\rho &= \rho_0\{1-\alpha(T-T_0)-\beta(T-T_0)^2\} \\ &= \rho_0\{1-\alpha(T_1-T_0)\theta-\beta(T_1-T_0)^2\theta^2\}\end{aligned} \tag{8}$$

in terms of the parameter

$$\theta = \frac{T-T_0}{T_1-T_0}. \tag{9}$$

Here ρ_0, α, β are constants, and T_0, T_1 are two representative absolute temperatures (suggested by the empirical formula (3) for water). Therefore, by the equation of continuity (5), one can define a solenoidal vector

$$\mathbf{q}_0 = \mathbf{q}\{1-\alpha(T_1-T_0)\theta-\beta(T_1-T_0)^2\theta^2\}$$

such that

$$\nabla.(\rho\mathbf{q}) = \rho_0\nabla.\mathbf{q}_0 = 0. \tag{10}$$

When converting to dimensionless variables, it is convenient to take a length unit d, and to write ∇ (as before) for the new dimensionless Laplacian operator. Also, let

$$\mathbf{g} = g\mathbf{g}' \tag{11}$$

where $\mathbf{g}'$ is a unit gravity vector in the dimensionless system, and let the dimensionless variables $\boldsymbol{\zeta}$, H, σ be defined by

$$(K_m/K_w)\boldsymbol{\zeta} = \mathbf{q}_0 d/\kappa, \tag{12}$$

$$(K_m/K_w)\nabla H = (\nabla p/\rho_0 - \mathbf{g})k/(\kappa\nu_0), \tag{13}$$

$$\sigma = \nu/\nu_0, \tag{14}$$

where $\kappa = K_w/\rho_0 c$ is the thermal diffusivity of the liquid, and ν_0 is the value of ν, both at $T = T_0$. From equation (2), η is given by

$$(K_m/K_w)\eta = \frac{k\alpha(T_1-T_0)gd}{\kappa\nu_0}. \tag{15}$$

For liquid flow through permeable material under moderately low pressures and at low velocities, as in near-surface ground-water movements, the work done by compression and viscous dissipation is assumed to be small. It follows that the second and third terms on the left-hand side of equation (7) may be neglected.

With the above definitions and approximations, it is found that substitution of (8) to (15) into (5) to (7) leads to the following dimensionless differential equations for liquid flow:

$$\nabla \cdot \boldsymbol{\zeta} = 0, \tag{16}$$

$$\nabla H + \eta \mathbf{g}'\{\theta + (\beta/\alpha)(T_1 - T_0)\theta^2\} + \sigma\boldsymbol{\zeta} = 0, \tag{17}$$

$$\boldsymbol{\zeta} \cdot \nabla\theta = \nabla^2\theta. \tag{18}$$

Equations (16), (17) and (18), together with the relation

$$\sigma \equiv \sigma(\theta), \tag{19}$$

constitute a system of six differential equations to determine the six unknowns θ, σ, H and the three components of $\boldsymbol{\zeta}$.

4. Free convection with plane flow

In this section it will be assumed that the predominant fluid motion is due to free thermal convection. Therefore, it is convenient to eliminate the pressure terms by taking the curl of (17). Some simplification can be introduced by assuming plane flow, for which one can write

$$\boldsymbol{\zeta} = -\nabla \times (\mathbf{j}\psi) \tag{20}$$

by virtue of (16), $\mathbf{j}$ being a dimensionless unit vector normal to the flow plane, and ψ a scalar stream function. Then, from (18) and (17) respectively, the appropriate equations of energy and motion can be written as

$$\nabla^2\theta = -[\nabla\theta, \nabla\psi, \mathbf{j}], \tag{21}$$

$$\nabla \cdot (\sigma\nabla\psi) = \eta\{1 + (2\beta/\alpha)(T_1 - T_0)\theta\}[\mathbf{g}', \nabla\theta, \mathbf{j}], \tag{22}$$

together with (19). (Use has been made here of the vector notation $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = [\mathbf{a}, \mathbf{b}, \mathbf{c}]$.)

From equation (4), it is clear that (19) can be written

$$\sigma(\theta) = \{1 + a(T_1 - T_0)\theta\}^{-1}, \tag{23}$$

where

$$a = 1/(T_0 - 260) \tag{24}$$

when the liquid is water.

On the boundary, it will be assumed, for the present, that θ obeys the inhomogeneous mixed conditions

$$\partial\theta/\partial u + C_1\theta = C_2, \qquad (\mathbf{r} = \mathbf{r}^s), \tag{25}$$

where $\mathbf{r}$ is a general position vector, $C_1(\mathbf{r}^s)$, $C_2(\mathbf{r}^s)$ are real functions of the boundary position vector $\mathbf{r}^s$, and $\partial/\partial u$ signifies differentiation normal to the boundary. Further, it will be assumed that ψ obeys the homogeneous conditions

$$\partial\psi/\partial u + C_3\psi = 0, \qquad (\mathbf{r} = \mathbf{r}^s), \tag{26}$$

where, usually, either $1/C_3(\mathbf{r}^s) = 0$ (Dirichlet conditions) or $C_3(\mathbf{r}^s) = 0$ (Neumann conditions). Solutions of (21), (22), using (23), with the boundary conditions (25) and (26), must be of such form that, as η tends to zero, ψ tends uniformly to zero everywhere, and θ tends uniformly to the steady-state conduction solution.

Although the stability of the flow patterns represented by these solutions is an important question, it is outside the scope of this discussion, which is restricted to certain specific solutions in the case of slow steady motion.

The relative importance of convection and conduction in transporting heat is indicated by the magnitude of the parameter η. When η is large and the convection component predominates, a distinct boundary layer could exist. The usual approximations are then valid. However, a treatment along these lines is not needed here, because fairly small values of η are expected in most geophysical situations.

When the latter condition that η is small applies, it is feasible to seek an approximate solution by means of perturbation expansions, as indicated previously. Let the dependent variables θ, ψ be expanded in the power series

$$\theta = \sum_{r=0}^{\infty} \theta_r \eta^r, \tag{27}$$

$$\psi = \sum_{s=1}^{\infty} \psi_s \eta^s, \tag{28}$$

for which non-zero radii of convergence exist throughout the region in which a solution is required. Then, using (23), it is possible to expand σ in the power series

$$\sigma = \sum_{t=0}^{\infty} \sigma_t \eta^t, \tag{29}$$

where

$$\sigma_0 = \{1 + a(T_1 - T_0)\theta_0\}^{-1} \tag{30}$$

and σ_t/σ_0 is equal to the cofactor of the $(1, t)$th element in the infinite triangular matrix

$$(i,j) = \left(\frac{\delta_{0,i-j} + a(T_1 - T_0)\theta_{i-j}}{1 + a(T_1 - T_0)\theta_0}\right), \tag{31}$$

in which use has been made of the Kronecker delta notation.

When (27), (28) and (29) are substituted into (21) and (22), and the successive coefficients of 1, η, η^2... are equated to zero, the following sets of equations result:

$$\nabla^2\theta_0 = 0, \tag{32}$$

and

$$\nabla^2\theta_n = -\sum_{s=1}^{n} [\nabla\theta_{n-s}, \nabla\psi_s, \mathbf{j}], \tag{33}$$

$$\begin{aligned}\frac{1}{\sigma_0}\nabla.(\sigma_0\nabla\psi_n) = {} & a(T_1 - T_0)\sum_{s=1}^{n-1}\sum_{r=0}^{s}\sigma_{s-r}\nabla\theta_r.\nabla\psi_{n-s} + [\mathbf{g}', \nabla\theta_{n-1}, \mathbf{j}] + \\ & + \{a + (2\beta/\alpha)\}(T_1 - T_0)\sum_{r=0}^{n-1}[\mathbf{g}', \theta_r\nabla\theta_{n-r-1}, \mathbf{j}] + \\ & + a(2\beta/\alpha)(T_1 - T_0)^2\sum_{t=0}^{n-1}\sum_{r=0}^{t}[\mathbf{g}', \theta_r\theta_{t-r}\nabla\theta_{n-t-1}, \mathbf{j}]\end{aligned} \tag{34}$$

for $n \geqslant 1$.

For the boundary conditions, it is evident from the substitution of (27), (28) into (25), (26) that

$$\partial\theta_n/\partial u + C_1\theta_n = \delta_{0,n}C_2, \qquad (\mathbf{r} = \mathbf{r}^s), \tag{35}$$

for all $n \geq 0$, and that

$$\partial\psi_n/\partial u + C_3\psi_n = 0, \qquad (\mathbf{r} = \mathbf{r}^s), \tag{36}$$

for $n \geq 1$.

With the above specifications, one can then define a pair of Green's functions $F(\mathbf{r}|\mathbf{r}_p)$, $G(\mathbf{r}|\mathbf{r}_p)$ (with position vectors $\mathbf{r}$, $\mathbf{r}_p$) according to the differential equations

$$\nabla^2 F = -4\pi\,\delta(\mathbf{r}-\mathbf{r}_p), \tag{37}$$

$$\frac{1}{\sigma_0}\nabla\,.\,(\sigma_0\nabla G) = -4\pi\,\delta(\mathbf{r}-\mathbf{r}_p), \tag{38}$$

which make use of the notation of the Dirac delta function on the right-hand side. These equations have the homogeneous boundary conditions

$$\partial F/\partial u + C_1 F = 0, \qquad (\mathbf{r} = \mathbf{r}^s), \tag{39}$$

$$\partial G/\partial u + C_3 G = 0, \qquad (\mathbf{r} = \mathbf{r}^s), \tag{40}$$

so that the formal solution for θ_0 is a surface integral, and the formal solutions for the θ_n, ψ_n $(n \geq 1)$ can be written down in terms of integrals over the volume V of saturated permeable material. These integrals are

$$\theta_0 = \frac{1}{4\pi}\oint C_2 F\,dS_p = -\frac{1}{4\pi}\oint (C_2/C_1)\nabla F\,.\,d\mathbf{S}_p, \tag{41}$$

and

$$\theta_n = -\int_V F\{\text{right-hand side of (33)}\}\,dV_p \tag{42}$$

$$\psi_n = -\int_V G\{\text{right-hand side of (34)}\}\,dV_p \tag{43}$$

for $n \geq 1$. Here dS_p, dV_p represent the surface and volume integration elements respectively. Comparison with (33) shows that the right-hand side of (42) involves solutions up to θ_{n-1}, ψ_n, and comparison with (34) shows that the right-hand side of (43) involves coefficients up to θ_{n-1}, ψ_{n-1}. Hence it becomes possible in principle to solve the linear equations (32) to (34), or the corresponding integral equations (41) to (43), in appropriate successive order for θ_0, ψ_1, θ_1, ψ_2, θ_2..., each in terms of the preceding coefficients.

5. Calculation of perturbation coefficients

The practicability of the above perturbation scheme is now tested by means of a numerical example. A two-dimensional case is considered, and iterative (relaxation-method) solutions are obtained for the differential equations (32) to (34) expressed in finite-difference form. Fairly coarse nets are employed, and only the first few coefficients of the series, up to ψ_4, θ_4, are calculated.

The choice of boundary conditions for the model has been influenced by a geological situation which appears to exist at one of the geothermally-active areas of Wairakei, New Zealand. In figure 1, *EG* represents the ground surface and *HAFG*, *DAB* represent sections of a sheet of ignimbrite (an igneous formation) faulted along the line *BF*. Non-fractured ignimbrite has the property of very low permeability, so that fluid flow may be ignored in this material. Above the ignimbrite, and bounded by *EFAD*, are formations which possess appreciable fluid permeability. These formations appear to be saturated with meteoric water which, although at very high temperature in the deeper regions, is held in liquid form by the high pressure. Overlying this, and close to the surface *EG*, is a layer of mudstone which, in the calculations here, has been assumed to be impermeable. The region below *BAH* is believed to be a reservoir of trapped stream or superheated water at a temperature of about 250° C, and some of this fluid may escape through the fault fissure at *A* into the upper permeable layer. The permeable region above *AD* has been drilled to a depth of 1000 m without encountering ignimbrite, so that the existence of the ignimbrite continuation *DAB* is not definitely established. While the angle of dip of the fault *BF* is uncertain, it probably lies in the range 75–90°, and has been taken to be 90° here.

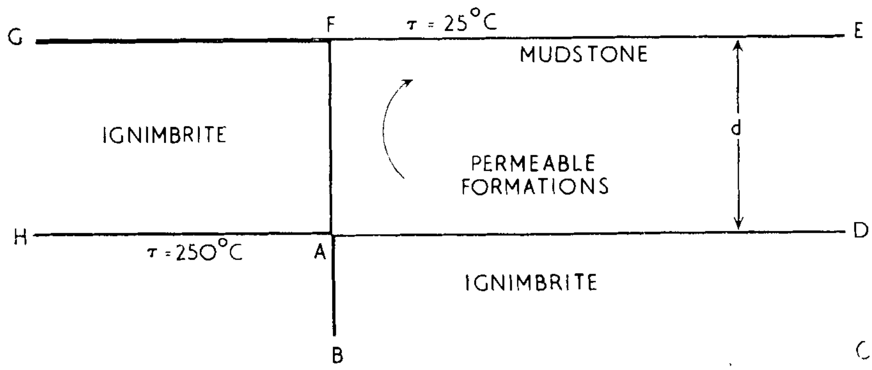

Figure 1. Hypothetical geological structure based upon a geological structure believed to exist at Wairakei, New Zealand.

Along *BAH* and *EG*, the temperatures are taken to be 250° C and 25° C respectively, giving $T_0 = 25+273 = 298°$ A, and $T_1 - T_0 = 225°$ A. The length unit d is taken as equal to the spacing between *AD* and *EF*. For the scale of this example, $d = O(10^5)$. Other values for parameters are obtained from equation (24), which gives $a = 1/38(°\,\mathrm{A})^{-1}$, and from a comparison of equations (3) and (8), which shows that $\alpha = 3{\cdot}17 \times 10^{-4}(°\,\mathrm{A})^{-1}$, $\beta = 2{\cdot}56 \times 10^{-6}(°\,\mathrm{A})^{-2}$. The boundary conditions are then as follows.

On *EG*: $\theta = 0$;

on *HA*, *AB*: $\theta = 1$;

on *BC*, *CE*, *GH*: $\partial\theta/\partial u = 0$, where $\partial/\partial u$ signifies the derivative normal to the boundary;

on *AD*, *DE*, *EF*, *FA*: $\psi = 0$.

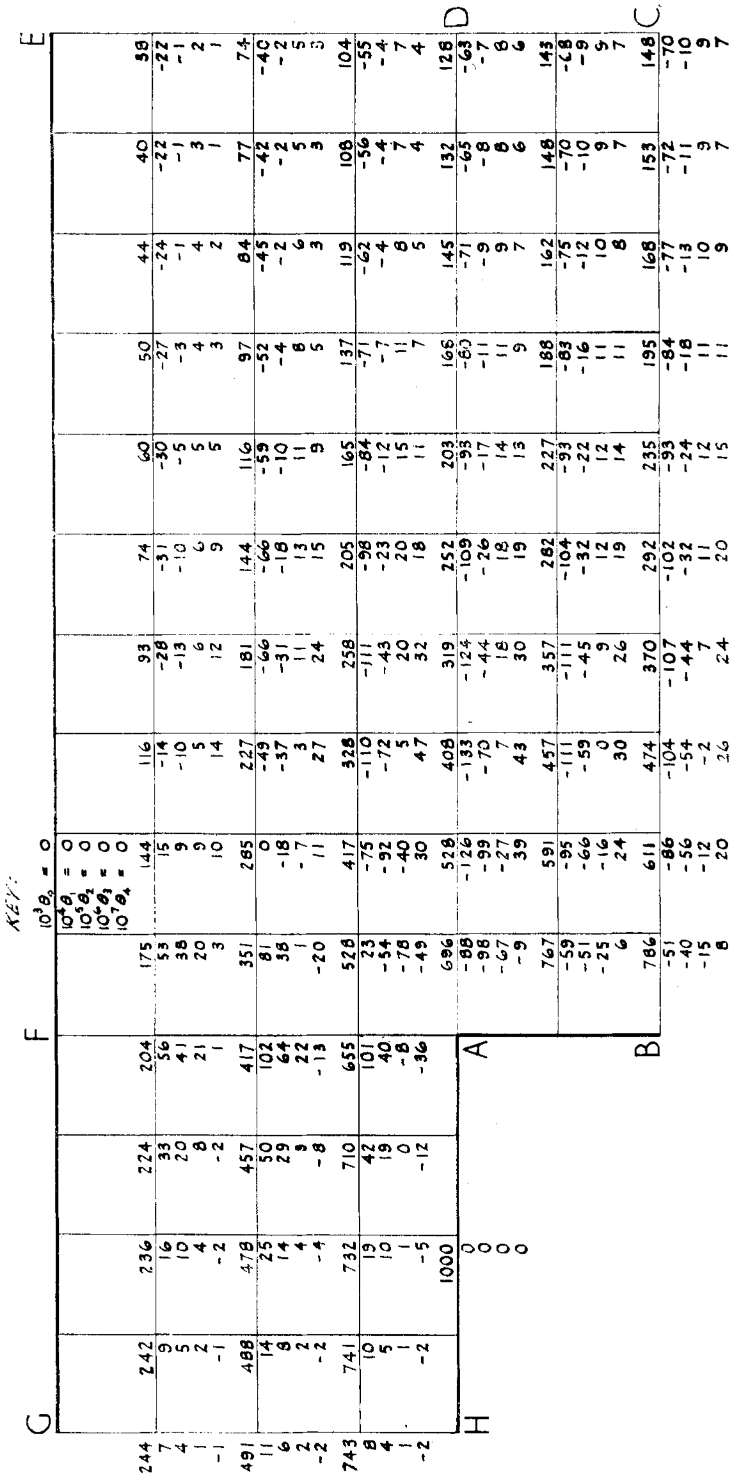

Figure 2. Numerical values obtained for the coefficients $\theta_0, \theta_1, \theta_2, \theta_3, \theta_4$ respectively, tabulated for each point of the relaxation net.

The condition $\partial\theta/\partial u = 0$ on BC is only an approximation to the true boundary condition. Also, it will be noted that fluid flow into the region $ADEF$ at A has been neglected.

For the region $ADEF$, the equations (21) and (22), when expressed in two-dimensional form, apply. However, for the regions $ABCD$, $HAFG$, ψ can be assumed to be identically zero, and θ obeys Laplace's equation.

In the absence of more precise information, the thermal conductivity has been assumed constant over the entire plane, so that at the boundaries FA, AD between differing media there is no discontinuity in θ or in its first normal derivative. As there are second-derivative discontinuities in θ, the isotherms show changes in curvature on the boundaries FA, AD.

Figure 2 gives the numerical values obtained for the finite-difference solutions for θ_0, $\theta_1 \dots \theta_4$, and figure 3 gives the values for $\psi_1 \dots \psi_4$. In the calculation of these solutions, several approximations were involved, as follows.

(*a*) A boundary singularity in θ exists at the point A, and partial compensation for its effect was introduced (Woods 1953) during the calculation of θ_0. As the modification was not large, the effect of the singularity was ignored during the calculation of higher θ-coefficients.

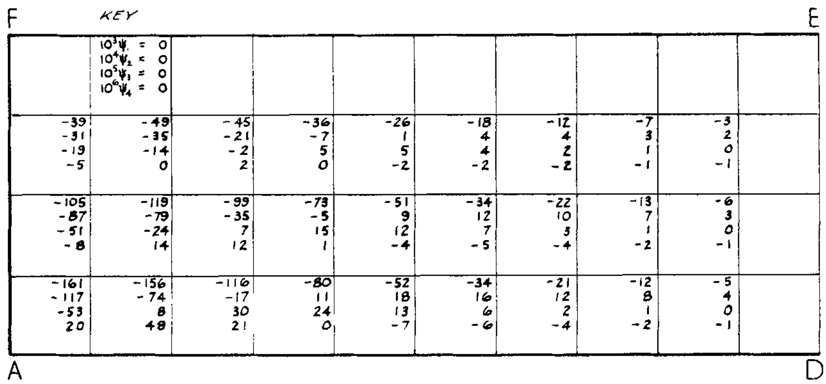

Figure 3. Numerical values obtained for the coefficients ψ_1, ψ_2, ψ_3, ψ_4 respectively, tabulated for each point of the relaxation net.

(*b*) On the boundaries FA and AD, the equations obeyed by the coefficients θ_n ($n \geqslant 1$) undergo a transition from (33) to Laplace's equation. The treatment of this type of singularity in a finite-difference solution involves taking half the contribution due to the right-hand side of (33) at mesh points lying on the discontinuity, and in refining the net close to the discontinuity. Since the refinement of the net has been omitted for this example, errors of approximation must be expected in the vicinity of the boundaries FA, AD.

(*c*) The calculation of ψ_3 and ψ_4 from (34) is approximate in that the terms under the first double-summation sign (which are of order n) on the right-hand side have been neglected in comparison with the other right-hand terms (which are of order $n-1$). The resultant errors introduced are negligible in the regions of highest temperature, rising to a few percent in the cold regions where σ is large and ψ is small.

Owing to the extremely high temperature differences applying on the boundaries, the magnitude of the coefficients does not decrease very rapidly, and it is necessary to consider the number of terms required to give reasonable accuracy, for a given value of the parameter η. From figures 2 and 3, there are indications that the sequences (θ_n) and (ψ_n) are oscillatory, whence a reasonably satisfactory estimate should be obtainable by choosing a suitable point to truncate each series, e.g. after a change in sign of the last coefficient.

Figure 4 is plotted from the finite-difference solution for θ_0, and corresponds to the temperature field due to conduction alone.

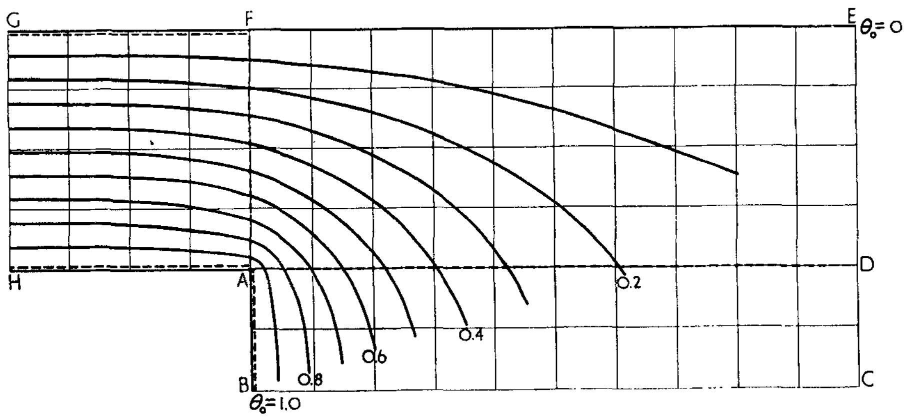

Figure 4. Field of the temperature parameter θ_0, corresponding to the conduction field in the geothermal model.

In figure 5, a value of $\eta = 7$ has been chosen to illustrate the alteration in the θ-field with convection present, the series being taken as far as the term in η^3. (The effect of adding the term in η^4 is shown by the broken lines.) Three departures from the conduction case may be noted: (1) a crowding of the isotherms towards point A from the right, corresponding to a decrease of temperature in the vicinity of AD; (2) an increase of temperature in the vicinity of FA; and (3) a marked horizontal temperature gradient as boundary AD is approached from above. If η were increased, these phenomena would be intensified, and the isotherms would tend to form a 'mushroom' situated to the right of the boundary FA.

Phenomena (1) and (2) have been observed in the data obtained from temperature measurements in boreholes at the Wairakei geothermal area, and comparison of the temperature gradients with theoretical results indicates that $\eta = O(10)$. However, (3) has not been observed, so that, although the presence of a convection process appears highly probable, there is no confirmation of the existence of the ignimbrite sheet *DAB*. There are indications, also, that the mudstone layer *EG* has appreciable permeability, perhaps due in part to erosion by surface streams.

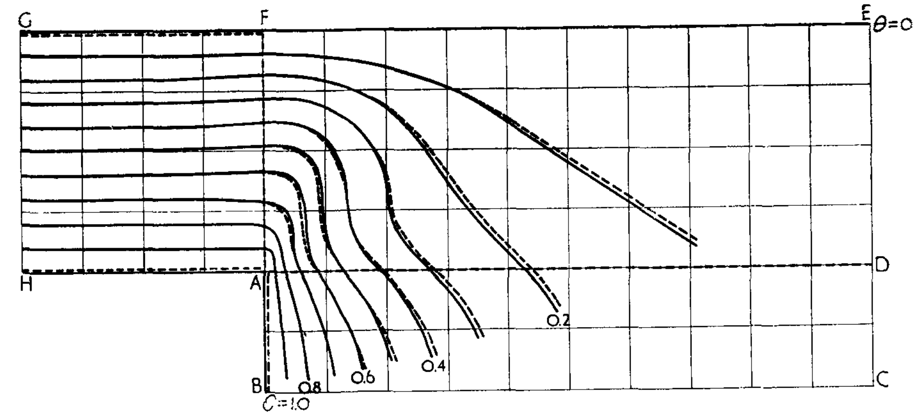

Figure 5. Field of the approximate temperature parameter θ defined by

$$\theta = \theta_0 + \theta_1 \eta + \theta_2 \eta^2 + \theta_3 \eta^3,$$

for $\eta = 7$. The broken-line isotherms indicate the modifications to the field when a further term $(\theta_4 \eta^4)$ is added to the series.

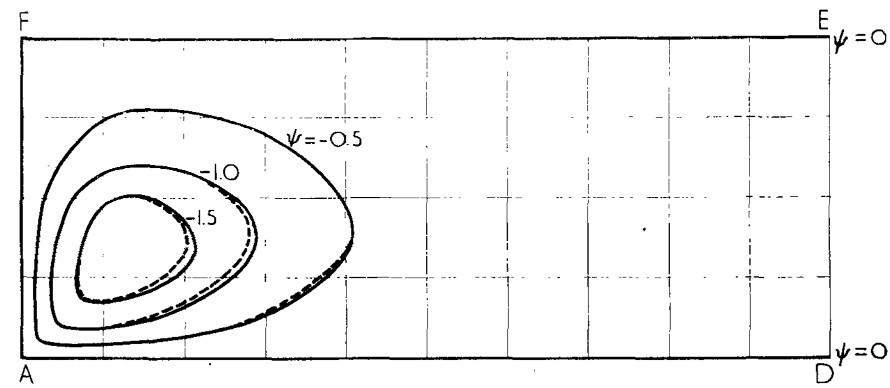

Figure 6. Field of the approximate stream function ψ defined by

$$\psi = \psi_1 \eta + \psi_2 \eta^2 + \psi_3 \eta^3,$$

for $\eta = 7$. The broken-line isopleths indicate the modifications to the field when a further term $(\psi_4 \eta^4)$ is added to the series.

In figure 6 are plotted values of the series for the stream function ψ, up to the η^3 term (full lines) and the η^4 term (broken lines), for $\eta = 7$. Modification by the fourth-order term is in the direction of decreasing ψ, which follows from the sign change in the higher coefficients. It will be noted that the region of main circulation is concentrated near the corner *A*, where a large temperature gradient exists, and where the high temperature results in a low fluid viscosity.

An estimate of the heat output Q at the surface EG (per unit width of boundary parallel to the axis of convection) for the boundary conditions given is expressible in dimensionless form in terms of the Nusselt number N:

$$N = \frac{Q}{K_m(T_1 - T_0)} = \int_{EG} (\partial\theta/\partial u)\, dl, \tag{44}$$

where dl represents an element of length of the boundary EG. Clearly, N is a function of η, and it is found that

$$N(\eta)/N(0) \doteqdot 1 - 0{\cdot}52 \times 10^{-3}\eta + 4{\cdot}25 \times 10^{-4}\eta^2 + 5{\cdot}05 \times 10^{-5}\eta^3 + 2{\cdot}76 \times 10^{-6}\eta^4, \tag{45}$$

$N(0)$ being the Nusselt number for conduction alone, with the given boundary temperatures. When $\eta = 7$, $N(\eta)/N(0) \doteqdot 1{\cdot}04$.

For the boundary conditions of this example, it is evident that the chosen value of 7 for the parameter η tests the perturbation method practically to the limit of its usefulness. A low maximum value of η is to be expected, since the temperature difference ($T_1 - T_0 = 225°$ A) is so large. Near 250° C, the density of saturated water is about 0·8 of the value at 25° C, and the kinematic viscosity is less than $\frac{1}{6}$ of the value at 25° C. Nevertheless, these extreme conditions are known to exist in at least one geothermal area, and indications are that the value of η appropriate to the area does not greatly exceed the value chosen in this example.

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